The interparticle interaction and noncommutativity of conjugate operators in quantum mechanics II. Lightest nuclei.

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A new model for calculating the structure of bound states of interacting particles is considered. The model takes into account the noncommutativity of the space and impulse operators plus the correlation equations for the indeterminacy of these quantities. The efficiency of the model is demonstrated by specific calculations for some lightest nuclei.

1. Introduction. The formulation of the model.

In our recent work [1] we have proposed a new quantum mechanical model for interacting bodies. The idea [2]- [3] that the coordinate and impulse operators for different particles may be not commutative make up a basis of the NOCE model (the NOCE model means the noncommutativity of the operators and the correlation equations). Within the framework of the NOCE model, we have examined [1] in detail the ground and some excited states of Hydrogen-like (H-like) atoms. In the present paper the ${}^2H \equiv D$,${}^3H$ and ${}^3He$ lightest nuclei are studied by a given model.

In the case of A identical particle the evident generalization of the equations (5)-(8) from the paper [1] yields the following relations

$$[x_k, \hat{p}_l^x] = [y_k, \hat{p}_l^y] = [z_k, \hat{p}_l^z] = \begin{cases} i\hbar\,\beta, & k = l; \\ i\hbar\,\beta_o, & k \neq l; \end{cases} \tag{1}$$

$$\hat{\mathbf{p}}_l = -i\hbar\,\beta \cdot \bigtriangledown_l, \qquad k, l = 1, 2, ..., A\,;$$

where there are only two noncommutativity parameters β and β_o present. Assuming the commutator of the operator of the particle coordinates and the operator of the total impulse $\mathbf{P}$ to be equal $i\hbar$ one obtains from the above mentioned equation a simple connection between β and β_o

$$1 = \beta + (A-1)\,\beta_o\,. \tag{2}$$

For this reason, only one additional equation is needed in order to find the noncommutativity parameters β and β_o. We write this equation by examining the process of the measurement of the coordinate of k-th particle with maximum accuracy. The reasoning similar to that employed in [1] leads us to the equation connecting the noncommutativity parameter β_o and the matrix element (ME) of the force $f_o \equiv \langle|F|\rangle$, exactly,

$$\beta_o = \frac{2\hbar c}{\varepsilon^2}\gamma_o \cdot f_o\,; \qquad \varepsilon \equiv m\,c^2\,. \tag{3}$$

Here m is the nucleon mass, and γ_o is one of the correlation factors specified by the equations

$$\Delta x_k\,\Delta p_l^x = \Delta y_k\,\Delta p_l^y = \Delta z_k\,\Delta p_l^z = \begin{cases} \frac{\hbar}{2\gamma}\cdot\beta, & k = l; \\ \\ \frac{\hbar}{2\gamma_o}\cdot\beta_o, & k \neq l; \end{cases} \tag{4}$$

$$k, l = 1, 2, ..., A.$$

Eqs.(2) and (3) yield one more useful relation :

$$\beta \;=\; 1 - \frac{2\hbar c}{\varepsilon^2}\cdot(A-1)\gamma_o f_o\,. \tag{5}$$

In the NOCE model, we have the analogy Schrödinger equation (SE)

$$\left[-\tfrac{\hbar^2}{2m'}\sum\limits_{i=1}^{A}\nabla_i^2 + \sum\limits_{i>j=1}^{A}\hat{V}(\mathbf{r}_{ij})\right]\Psi_E(1,2,...,A) = E\,\Psi_E\,,$$

$$m^{'} = m/\left[1 - \tfrac{2\hbar c}{\varepsilon^2}\cdot(A-1)\gamma_o f_o\right]^2. \tag{6}$$

Specific solutions to these equations can be found by the method of successive iterations, where at the 1-st step the conventional SE with the masses $m^{'} = m$, which the particles have in the absence of the interaction ($\beta \equiv 1$), has to be solved. After that, on finding the wave function ψ, one can calculate ME of the force $\langle|F|\rangle$ and the first value of the commutation parameter β distinct from unity. At the 2-nd step, SE is solved with the modified particle masses $m^{'} = m/\beta^2$. On finding the new ψ, we calculate the quantity $\langle|F|\rangle$ and compare it with the one obtained at the 1st step. Then, we proceed with the iterations until the values of the matrix element of the force $\langle|F|\rangle$ obtained at subsequent steps will be virtually indistinguishable. It is clear that, before starting the above iteration process, we should specify the numerical value for the correlation factor γ_o entering Eqs.(6). To calculate its, one can employ specific parameters of a given system based on reliable experimental data. The way to practically implement this will be described in detail hereinafter

2. Deuteron and the variational Ritz principle.

Now we proceed to consideration of nuclear systems taking as an example the simplest one, the deuteron. As is known, in contrast to the atom theory, in the nuclear theory one has to deal with a serious problem of choosing the nucleon-nucleon (NN) forces. In view of this circumstance the accuracy of theoretical estimates for a nucleus is considerably lower than that of similar calculations in atomic physics. For this reason, taking into account the calculations of H-atoms given in [1], such a small correction in relation to the results of conventional SE may evidence, at first sight, the inefficiency of application of the NOCE model to nuclei. However, if it is remembered that the matrix element (ME) of the force $\langle|F|\rangle$ for nucleon systems should be several orders greater than similar values for $\langle|F|\rangle$ in atoms, the contrary anxiety arises, meaning that the application of the NOCE model to nuclear physics may result in definitely unrealistic theoretical estimates.

Apparently, to clarify these questions, one needs to perform specific numerical calculations. First, we can take as an example the deuteron with a simple central potential in the Yukawa form

$$V(r) = -V_o \frac{e^{-r/a}}{r/a}\,. \tag{7}$$

The bound states of the deuteron are described by the wave function $\psi(\mathbf{r}) = \frac{1}{r}\chi_{nl}(r)Y_{ln}(\theta,\varphi)$ depending on the vector of the relative motion $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Let us restrict ourselves to considering the ground s-state ($l = 0$), then, the radial wave function $\chi_{n=1,l=0}(r) \equiv \chi(r)$ has to satisfy the simple equation

$$\left[-\frac{\hbar^2}{2\mu'}\frac{d^2}{dr^2} + V(r) - E\right]\chi(r) = 0\,, \tag{8}$$

where the reduced mass μ' modified in the NOCE model equals

$$\mu' = \frac{m_1' m_2'}{m_1' + m_2'} = \frac{m}{2\left(1 - \frac{2\hbar c}{\varepsilon^2}\gamma_o\langle|F|\rangle\right)^2}; \quad \varepsilon = mc^2\,. \tag{9}$$

To write the equation (9), we use the relation

$$m_{proton} = m_{neutron} \equiv m\,, \tag{10}$$

wich are natural for the case of identical particles. Satisfactory solutions to the equation (8) can be found by means of the variational Ritz principle. For the Yukawa-type potentials, good results can be obtained by using the function

$$\chi(r) = 2\eta^{3/2} r e^{-\eta r} \tag{11}$$

as the variational one. Here the only variational parameter η has to be found from the condition that the deuteron energy

$$E(\eta) = \frac{\hbar^2}{2\mu'}\,\eta^2 - V_o\,\frac{4(a\,\eta)^3}{(1+2a\,\eta)^2} \tag{12}$$

must take extremum. In what follows, it is convenient, following Ref. [4], to introduce the notation

$$K \equiv \frac{2\mu' a^2}{\hbar^2} V_o\,, \quad p \equiv 2a\,\eta\,. \tag{13}$$

Then the above mentioned condition $dE/d\eta = 0$ yields the cubic equation with respect to the quantity p

$$p^3 + (3-K)\,p^2 + 3(1-K)\,p + 1 = 0\,. \tag{14}$$

As is evidenced by the immediate calculation, the optimal value is associated with the root p_o, which can be expressed via the quantity K introduced in (13) as

$$p_o = \frac{K}{3} - 1 + 2\sqrt{\frac{K}{3}\left(1+\frac{K}{3}\right)}\cdot\cos\left\{\frac{\pi}{3} - \frac{1}{3}\arccos\left[\frac{K - \frac{K^2}{6} - \frac{K^3}{27}}{(\frac{K}{3}+\frac{K^2}{9})^{3/2}}\right]\right\}\,. \tag{15}$$

The matrix element of the force

$$F(r) = -\frac{d}{dr}V(r) = -a\,V_o\left(\frac{1}{r^2} + \frac{1}{ar}\right)\,\exp(-\frac{r}{a}) \tag{16}$$

on the functions (11) is equal

$$f_o \equiv \langle\chi(r)|F(r)|\chi(r)\rangle = 8\frac{V_o}{a}(a\,\eta)^3\frac{(1+a\,\eta)}{(1+2a\,\eta)^2} = \frac{V_o}{a}\cdot p^3\frac{(1+\frac{p}{2})}{(1+p)^2}\,. \tag{17}$$

Substituting the value of the root p_o from (61) instead of p into the relation (17) and taking into account that the quantity

$$K = \frac{a^2 V_o}{(\hbar c)^2}\cdot\frac{\varepsilon}{(1-\frac{2\hbar c}{\varepsilon^2}\gamma_o\,f_o)^2} \tag{18}$$

is, in turn, expressed via f_o, we obtain some rather complicated equation for ME of the force f_o. The equation found in this way plays, apparently, the same role as the equation (33) from [1] does in the case of H-like atoms, namely, solving this equation enables us to avoid the complicated (in computational respect) procedure of successive iterations. The necessary solutions have been found in the graphical way. In doing so, from the set of all the solutions found (i.e., the points of intersections of the straight line $y = x \equiv f_o$ with the plot of the function $y = \varphi_2(x)$ representing

the right-hand side of the equation (17)) we selected the one associated with the minimum value of $f_o > 0$.

In the numerical calculations performed here, we used the parameters of the potential (7) given in [5], $V_o = 20.7$ MeV, $a = 2.43$ fm. This potential was adjusted to fit the experimental data on the scattering length and effective radius of the triplet state in the "proton + neutron" system. The nucleon mass was equal $\varepsilon \equiv mc^2 = 931.441$ MeV. The calculations were carried out by using the various allowed ($[0 < \gamma_o \leq 1]$) values of the correlation factor γ_o. The results obtained are given in the Table 1. In this table, the ground state energy of the deuteron $E = E_D$, ME of the force f_o, the optimum value of the root $p_o \equiv 2a\,\eta_o$, and the values of the quantities

$$1 - 2\,x_o \equiv \beta, \quad 2\,x_o \equiv \beta_o; \quad x_o \equiv \gamma_o \frac{\hbar c}{\varepsilon^2}\, f_o\,, \tag{19}$$

defining the commutation relations (1) are presented as dependent on the quantity γ_o.

As is seen from the table, the binding energy of the deuteron $|E_D|$ (the experimental value $|E_D^{exp.}| = 2.22457$ MeV) grows with γ_o from $|E_D| = 2.2019$ MeV at $\gamma_o = 0.00001$ up to $|E_D| = 2.3826$ at $\gamma_o = 1$. This means that in the NOCE model under consideration, for the potential of NN-forces chosen, the maximum possible increase of $|E_D|$ constitutes $\sim 8\%$ as compared to the calculation of $|E_D|$ based on the conventional SE. With increasing the γ_o parameter from 0.00001 to 1 the quantity β diminishes insignificantly (by less than 1%) while the quantity β_o grows from $\approx 0.906 \cdot 10^{-7}$ at $\gamma_o = 0.00001$ to $\beta_o \approx 0.01$ at $\gamma_o = 1$.

The experimental binding energy of the deuteron corresponds to the values $\gamma_o = 0.13523$, $f_o = 20.0812 MeV/fm$, and $\beta = 0.998\,765$, accompanied by the increase in the nucleon mass by the value of $\triangle m = (\frac{1}{\beta^2} - 1)\, m = 0.00247\,516\, m$.

3. Mirror nuclei ^3H and ^3He.

In this section, we consider the results of the calculations of the ground states for the Tritium and Helium-3 nuclei.

In the case of a central NN-potential $V(r_{ij})$, the force F entering the definition for the quantity f_o is equal to the sum of derivatives of $V(r_{ij})$ with respect to the relative distance r_{ij}, i.e.

$$F = -\sum_{i>j=1}^{A} \frac{d}{dr_{ij}} V(r_{ij})\,, \qquad r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|\,. \tag{20}$$

In specific calculations of light nuclei, one frequently restricts consideration to the case of the central exchange potential

$$V(r_{ij}) = -\sum_{S,T=0,1} V_{2S+1,2T+1}(r_{ij})\,\hat{P}_{2S+1,2T+1}(ij)\,, \tag{21}$$

where the known projection operators $\hat{P}_{2S+1,2T+1}$ do "cut" the relevant spin-isospin states of the interacting nucleon (ij)-pair from the wave function $\Psi(1,2,...,A)$. The radial dependence of the components of NN-potential $V_{2S+1,2T+1}(r_{ij})$ can be presented, without loss of generality, in the form of a series in Gaussian terms

$$V_{2S+1,2T+1}(r_{ij}) = \sum_{\nu=1}^{\nu_{pot}} V^{[\nu]}_{2S+1,2T+1} \cdot \exp\left(-\frac{r_{ij}^2}{\mu_\nu^2}\right)\,. \tag{22}$$

Now we consider the problem of the bound states for the nuclei ^3H and ^3He. In the NOCE model, the conventional three-body SE

$$\hat{H}\,\Psi(1,2,3) = E\,\Psi, \quad \hat{H} = \sum_{i=1}^{3}\left(-\frac{\hbar^2}{2m'}\right)\nabla_i^2 + \frac{Ze^2}{|\mathbf{r}_1-\mathbf{r}_2|} + \sum_{i>j=1}^{3} V(|\mathbf{r}_1-\mathbf{r}_2|)\,, \tag{23}$$

with the modified nucleon mass

$$m' = \frac{m}{\beta^2}, \qquad \beta = 1 - \frac{4\hbar c}{\varepsilon^2}\,\gamma_o\, f_o\,. \tag{24}$$

is valid. The charge entering (23) is $Z = 0$ for ^3H nucleus (1-st and 2-nd particles represent neutrons), and $Z = 1$ for the nucleus ^3He (1-st and 2-nd particles are protons), respectively.

In specific investigations of the structure of light nuclei, we use as the NN-potential of the type (21)-(22) the two-Gaussian Volkov potential [6] with the following parameters (I-st version), 1-st component ($\nu = 1$) $V_{31}^{[\nu=1]} = V_{13}^{[\nu=1]} = 144.86$ MeV , $V_{33}^{[\nu=1]} = V_{11}^{[\nu=1]} = -28.972$ MeV, μ_1=0.82 fm; 2-nd component ($\nu = 2$) $V_{31}^{[\nu=2]} = V_{13}^{[\nu=2]} = -83.34$ MeV, $V_{33}^{[\nu=2]} = V_{11}^{[\nu=2]} = 16.668$ MeV, μ_2=1.60 fm.

In view of modern state of computer technology, it is convenient to seek the solution of three-body SE by using the variational Ritz principle with the wave function of the system being expanded in a series of the known basis functions. These latter may contain one or another set of variational parameters, which, actually, determines the flexibility of the basis chosen. In the case of the lightest nuclei, satisfactory results were obtained by using the simplest Gaussian-type functions. The efficiency of using the Gaussian basis of functions in few-body problems has been first realized by the theorists of the Moscow State university in 1973-1975 (see, for instance, [7] - [10]). Really, even the first calculations performed within the

framework of the stochastic variational method (SVM) gave the results of highest accuracy possible at that time. In view of this, some authors sometimes refer to these calculations as the "high precision" ones. Nowadays, the SVM method is widely and successfully employed in theoretical studies of various quantum systems consisting of small number of particles [11] - [13].

That is why, it is reasonable to employ here SVM to calculate the lightest nuclei under consideration. In doing so, the total wave function ^3H and ^3He is presented, as usually, in the form of a product of the space and spin-isospin function

$$\Psi(1,2,3) = \psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \cdot \chi(1,2,3)\,. \tag{25}$$

According to the experimental situation, the function χ represents the determinant

$$\chi(1,2,3) = \frac{1}{\sqrt{3!}} \begin{vmatrix} \chi_{\uparrow,\uparrow}(1) & \chi_{\uparrow,\uparrow}(2) & \chi_{\uparrow,\uparrow}(3) \\ \chi_{\downarrow,\uparrow}(1) & \chi_{\downarrow,\uparrow}(2) & \chi_{\downarrow,\uparrow}(3) \\ \chi_{\uparrow,\downarrow}(1) & \chi_{\uparrow,\downarrow}(2) & \chi_{\uparrow,\downarrow}(3) \end{vmatrix}, \tag{26}$$

composed by the one-particle spin-isospin functions

$$\chi_{\sigma\tau}(i) = \chi_{\frac{1}{2},m_\sigma}(i)\chi_{\frac{1}{2},m_\tau}(i) \equiv \chi_{m_\sigma,m_\tau}(i)\,, \tag{27}$$

where the lower subscript m_σ of the function (27) takes two values denoted in (26) by an arrow "up" $\uparrow$ or "down" $\downarrow$. Exactly same notation is used for the other quantum number m_τ. Resolving the determinant makes it evident that the function $\chi(1,2,3)$ is antisymmetric with respect to permutations of spin-isospin coordinates of 1-st and 2-nd particles. For this reason, by virtue of Pauli principle, the ψ-function must be symmetric with respect to permutations of space coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$, which means, in turn, that the expansion of ψ-function should be carried out over the properly symmetrized Gaussian terms, i.e.,

$$\begin{aligned} &\psi_{L=0}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \sum_{j=1}^{j_{max}} C_j \cdot u_j(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)\,, \\ &u_j(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \hat{A}\,|j\rangle \equiv \hat{A} \exp\{-\alpha_{12}^j\, r_{12}^2 - \alpha_{13}^j\, r_{13}^2 - \alpha_{23}^j\, r_{23}^2\} = \\ &= \sum_{sym=1}^{2} \exp\left\{-\alpha_{12}^j(sym) \cdot r_{12}^2 - \alpha_{13}^j(sym) \cdot r_{13}^2 - \alpha_{23}^j(sym) \cdot r_{23}^2\right\}\,. \end{aligned} \tag{28}$$

The action of the symmetrization operator $\hat{A}$ on the usual Gaussian term $|j\rangle$ gives rise to a sum consisting, in this specific case, of two summands only with parameters α_{kl}^j being replaced by $\alpha_{kl}^j(sym)$. The result of the immediate examination of permutations associated with the operator $\hat{A}$ is given in the Table 2.

As mentioned above, in numerical calculations we used the central exchange Volkov potential, which contains the dependence on the spin-isospin coordinates in the projection operators only. This circumstance makes it possible to easily calculate the matrix elements (ME) of the operator of NN-forces (21) with the spin-isospin functions. In particular, for Volkov potential in the case of the nuclei ^3H and ^3He under consideration, we obtain

$$\langle\chi(1,2,3)\left|\sum_{i>j=1}^{3}V(r_{ij})\right|\chi(1,2,3)\rangle = \tag{29}$$

$$= V_{13}(r_{12}) + \tfrac{1}{4}\left[3V_{31}(r_{13}) + V_{11}(r_{13}) + 3V_{31}(r_{23}) + V_{11}(r_{23})\right] .$$

Now, in order to write the system of linear equations for the expansion coefficients of the wave function C_j, we calculate the matrix elements of all the operators entering the definition for the Hamiltonian (23) on the basis function $u_j(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$. Earlier, the analytical calculations of this kind have been performed in the number of works (some of them are cited above). In view of this, we cite here only selected results sticking to the notation of the paper by N.N.Kolesnikov [10]. Let us begin with the overlap integral for the basis functions :

$$\langle u_{j'}|u_j\rangle = \sum_{zym=1}^{2}\sum_{sym=1}^{2}\langle j^{'},zym|j,sym\rangle; \qquad \langle j^{'},zym|j,sym\rangle = \left(\frac{\pi}{\sqrt{D^{[j',j]}}}\right)^3, \tag{30}$$

with the notation

$$\begin{aligned} &|j,sym\rangle \equiv \exp\left\{-\alpha_{12}^{j}(sym)\cdot r_{12}^2 - \alpha_{13}^{j}(sym)\cdot r_{13}^2 - \alpha_{23}^{j}(sym)\cdot r_{23}^2\right\} ; \\ &D^{[j',j]} \equiv \alpha_{12}\alpha_{13} + \alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{23}; \quad \alpha_{kl} \equiv \alpha_{kl}^{j}(sym) + \alpha_{kl}^{j'}(zym) . \end{aligned} \tag{31}$$

It is clear that ME for any other operator on the functions (28) represents a sum of the type (30), i.e, the analytical calculations reduce, actually, to finding the partial ME with the functions $|j,sym\rangle$. For the operator of the kinetic energy, we obtain

$$\begin{aligned} &\langle j^{'},zym|\hat{T}|j,sym\rangle = \tfrac{3\hbar^2}{2m'}\tfrac{\langle j^{'},zym|j,sym\rangle}{D^{[j',j]}}\cdot \\ &\cdot\sum_{k=1}^{3}\sum_{l',l=1}^{3}\alpha_{kl}^{j'}(zym)\alpha_{kl}^{j}(sym)\left[D_{l'k}^{[j'j]} + D_{lk}^{[j'j]} - D_{l'l}^{[j'j]}\right]; \qquad (l',l\neq k), \end{aligned} \tag{32}$$

with the notation [10]

$$\begin{aligned} &D_{lk}^{[j'j]} \equiv \tfrac{\partial}{\partial\alpha_{kl}}D^{[j'j]}, \quad D_{ll}^{[j'j]} = 0; \\ &D_{12}^{[j'j]} = \alpha_{13}+\alpha_{23}, \quad D_{13}^{[j'j]} = \alpha_{12}+\alpha_{23}, \quad D_{23}^{[j'j]} = \alpha_{12}+\alpha_{13} . \end{aligned} \tag{33}$$

The calculation of ME of the operator of potential energy is associated with the expression

$$\langle j^{'}, zym|V(r_{kl})|j, sym\rangle = \sum_{\nu=1}^{\nu_{pot}} V^{[\nu]} \left(\frac{\pi}{\sqrt{D^{[j^{'}j]} + \frac{1}{\mu_\nu^2} D_{kl}^{[j^{'}j]}}} \right)^3 , \qquad (l^{'}, l \neq k), \tag{34}$$

where

$$V(r_{kl}) = \sum_{\nu=1}^{\nu_{pot}} V^{[\nu]} \cdot \exp\left(-\frac{r_{kl}^2}{\mu_\nu^2}\right) \tag{35}$$

is one of the components of the NN-potential (21) remained in (29) after averaging over spin-isospin functions $\chi(1,2,3)$. When considering the nucleus ^3He, we have to calculate also the Coulomb ME

$$\langle j^{'}, zym \left| \frac{e^2}{r_{kl}} \right| j, sym\rangle = \frac{2e^2}{\sqrt{\pi}} \langle j^{'}, zym|j, sym\rangle \sqrt{\frac{D^{[j^{'}j]}}{D_{kl}^{[j^{'}j]}}} . \tag{36}$$

Now, we have all the relations needed to write the system of equations

$$H \cdot X = \lambda B \cdot X , \tag{37}$$

which is necessary to find the energy spectrum of the nucleus $\{E_i\} = \lambda$ and the relevant coefficients in the expansion $\{C_j\} = X$ of the space wave function $\psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$. According to the definition of the matrix elements $B \equiv ||B_{j^{'}j}||$ and $H \equiv ||H_{j^{'}j}||$ in Eq.(37), we obtain

$$\begin{aligned} B_{j^{'}j} &\equiv \langle u_{j^{'}}|u_j\rangle = \sum_{zym=1}^{2} \sum_{sym=1}^{2} \langle j^{'}, zym|j, sym\rangle , \\ H_{j^{'}j} &= \sum_{zym=1}^{2} \sum_{sym=1}^{2} \langle j^{'}, zym|H = T + U + U_{Coul.}|j, sym\rangle . \end{aligned} \tag{38}$$

Thus, the energy of three nuclei is sought as a result of solving, in the first place, the generalized problem for eigenvalues, and, in the second place, the problem of finding the optimum values for the variational parameters α_{kl}^j determining the basis functions (28). In so doing, the most difficult procedure herewith is the optimization. In performing numerical calculations, we employed the known mathematical library $IMSL$, specifically, we used the subroutine $DGVCSP$ [14] in the generalized problem for eigenvalues, and the subroutine $DBCONF$ [15] for the optimization procedure.

It should be kept in mind that the above mentioned calculations are carried out only after the value of the nucleon mass $m^{'} = m/\beta^2$ is found. The latter is

determined by the magnitude of ME of the force f_o (see the relation (24)). In this

$$f_o = \langle \Psi(1,2,3) \left| \sum_{i>j=1}^{3} -\frac{dV(r_{ij})}{dr_{ij}} \right| \Psi(1,2,3) \rangle = \tag{39}$$

$$= \frac{1}{\langle \Psi(1,2,3) | \Psi(1,2,3) \rangle} \sum_{j',j=1}^{j_{max}} C_{j'} C_j \sum_{\nu=1}^{\nu_{pot}} \{ V_{13}^{[\nu]} \cdot G_{\nu}^{[j',j]}(1,2) +$$

$$+ \frac{1}{4}(3V_{31}^{[\nu]} + V_{11}^{[\nu]}) \cdot G_{\nu}^{[j',j]}(1,3) + \frac{1}{4}(3V_{31}^{[\nu]} + V_{11}^{[\nu]}) \cdot G_{\nu}^{[j',j]}(2,3) \} ,$$

with the partial ME being

$$G_{\nu}^{[j',j]}(k,l) \equiv \langle j', zym \left| \frac{2r_{kl}}{\mu_{\nu}^2} e^{-\frac{r_{kl}^2}{\mu_{\nu}^2}} \right| j, sym \rangle = \tag{40}$$

$$= \frac{4\pi}{\mu_{\nu}^2} \left(\frac{\pi}{D_{lk}^{[j'j]}} \right)^{3/2} \left\{ \frac{D_{lk}^{[j'j]}}{D^{[j',j]} + \frac{1}{\mu^2} D_{lk}^{[j'j]}} \right\}^2 .$$

The quantities $D^{[j',j]}$ and $D_{lk}^{[j'j]}$ entering this equation have been already specified by the relations (31) and (33), respectively. On calculating ME of the force f_o in this way, we evaluate the mass m' by Eq.(24) and proceed to solving SE with the modified mass m', i.e., we carry out the process of successive iterations described above (Section I). In so doing, we may put $f_o = 0 \quad (m' = m)$ at the first step, then, the 1-st iteration is identical to solving the conventional SE.

Consider now the calculations of the nuclei ^3H and ^3He carried out for Volkov NN-potential [6] with the model parameter $\gamma_o = 0,6$. In this calculation, the 15 functions (28) were employed. As is seen from the definition (28), each of these latter contains three independent variational parameters α_{12}^j, α_{13}^j and α_{23}^j. Some of the results of the calculations performed here are given in the Table 3. The values of the energy and ME of the force f_o are given in the units of MeV and MeV/fm, respectively. Since the NN-potential used in calculations was adjusted to fit the basic properties of light nuclei, the calculated binding energy $E^{Q.mech.}$ turned out to be close to the experimental value. As one would expect, the NOCE model overbinds the nuclei ^3H and ^3He to some extent (see Table 3). The energy gain constitutes $\sim 4\%$ as compared to $E^{Q.mech.}$, and the change in the nucleon mass is $\sim 0.9\%$.

It is clear that these numbers directly depend on a specific choice of the NN-potential, as well as on the magnitude of the parameter γ_o. For this reason, the theoretical results presented here are qualitative in nature, in contrast to atomic calculations [1]. However, they are sufficient to make estimates of the efficiency of the NOCE model under consideration. In others words, the calculation performed

made possible the estimation of the order of the expected corrections to the basic nuclear properties arising due to the allowance for the noncommutativity of the coordinate and impulse operators of the interacting particles.

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Table 1: The properties of the ground state of the deuteron within the NOCE model.

γ_o	E_D; (MeV)	f_o; (MeV/fm)	$p_o = 2a\,\eta$	β	β_o
0.00001	-2.20194412	19.9189032	2.27127197	0.99999991	$9.06148{\cdot}10^{-8}$
0.01	-2.20360006	19.9307771	2.27190365	0.99990934	$9.06688{\cdot}10^{-5}$
0.10	-2.21862818	20.038557	2.27762869	0.99908847	$9.12591{\cdot}10^{-4}$
0.20	-2.23559453	20.160273	2.28407581	0.99816586	$1.83426{\cdot}10^{-3}$
0.30	-2.25285174	20.284119	2.29061592	0.99723189	$2.76829{\cdot}10^{-3}$
0.40	-2.27040879	20.410160	2.29725172	0.99628626	$3.71398{\cdot}10^{-3}$
0.50	-2.28827498	20.538467	2.30398603	0.99532864	$4.671666{\cdot}10^{-3}$
0.60	-2.30646017	20.669115	2.31082180	0.99435870	$5.64166{\cdot}10^{-3}$
0.70	-2.32497459	20.802182	2.31776210	0.99337612	$6.62431{\cdot}10^{-3}$
0.80	-2.34382903	20.937744	2.32481016	0.99238052	$7.61997{\cdot}10^{-3}$
0.90	-2.36303480	21.075893	2.33196937	0.99137152	$8.62903{\cdot}10^{-3}$
1.00	-2.38260376	21.216715	2.33924329	0.99034874	$9.90348{\cdot}10^{-3}$

Table 2: Symmetrized variational parameters $\{\alpha^j_{kl}(sym)\}$ for the nuclei ^3H and ^3He.

$sym \quad \rightarrow$	1	2
$\alpha^j_{12}(sym)$	α^j_{12}	α^j_{13}
$\alpha^j_{13}(sym)$	α^j_{13}	α^j_{12}
$\alpha^j_{23}(sym)$	α^j_{23}	α^j_{23}

Table 3: The properties of the ground states of the ^3H and ^3He in the case $\gamma_o = 0.6$ and the Volkov potential.

	$T \equiv^3 H$	^3He
$E^{exp.}$; MeV	-8.482	-7.718
$E^{Q.mech.}$; MeV	-8.464	-7.759
$E^{theor.}$; MeV	-8.819	-8.113
f_o; MeV/fm	8.2483085	8.4056275
β	0.995498	0.995412
β_o	0.00225124	0.00229418